CHAPTER - 7 –APPLICATIONS OF DIFFERENTIAL CALCULUS

I. Answer the Following (2 marks)

1. Find the equation of the tangent and normal at any point to the Lissajous curve given by $x = 2\cos 3t$ and $y = 3\sin 2t$, $t \in \mathbb{R}$.
2. Find the slope of the tangent to the following curves at the respective given points.
   (i) $y = x^4 + 2x^2 - x$ at $x = 1$ (ii) $x = a\cos^3 t$, $y = b\sin^3 t$ at $t = \frac{\pi}{2}$.
3. Find the points on the curve $y^2 - 4xy = x^2 + 5$ for which the tangent is horizontal.
4. Evaluate the following limits, if necessary use l'Hôpital Rule (i) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ (ii) $\lim_{x\to\infty} \frac{2x^2-3}{x^2-5x+3}$
5. Find the absolute extrema of the following functions on the given closed interval.
   (i) $f(x) = x^2 - 12x + 10$; $[1, 2]$ (ii) $f(x) = 2\cos x + \sin 2x$; $\left[0, \frac{\pi}{2}\right]$
6. Find the local extrema for the following functions using second derivative test : $f(x) = x^2 e^{-2x}$

II. Answer the Following (3 Marks)

1. Find the angle between the curves $y = x^2$ and $x = y^2$ at their points of intersection (0,0) and (1,1).
2. Find the point on the curve $y = x^2 - x + 5\ 4$ at which the tangent is parallel to the line $3x + y = 7$.
3. Find the points on the curve $y = x^3 - 6x^2 + x + 3$ where the normal is parallel to the line $x + y = 1729$.
4. Find the tangent and normal to the following curves at the given points on the curve.
   (i) $y = x^2 - x^4$ at (1,0) (ii) $y = x\sin x$ at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$
5. Compute the value of 'c ' satisfied by Rolle's theorem for the function $f(x) = \log\left(\frac{x^2+6}{5x}\right)$ in the interval [2, 3].
6. Using the Lagrange's mean value theorem determine the values of x at which the tangent is parallel to the secant line at the end points of the given interval: (i) $f(x) = x^3 - 3x + 2$, $x \in [-2, 2]$ (ii) $f(x) = |3x + 1|$, $x \in [-1, 3]$
7. Evaluate the following limits, if necessary use l'Hôpital Rule : (i) $\lim_{x\to 0}\left(\frac{1}{\sin x} * \frac{1}{x}\right)$ (ii) $\lim_{x\to 1^+}\left(\frac{2}{x^2-1} - \frac{x}{x-1}\right)$
8. Find the intervals of monotonicities and hence find the local extremum for the following functions:
   (i) $f(x) = \frac{e^x}{1-e^x}$ (ii) $f(x) = \frac{x^3}{3} - \log x$
9. Find intervals of concavity and points of inflexion for the following functions:
   (i) $f(x) = x(x-4)^3$ (ii) $f(x) = \frac{1}{2}(e^x - e^{-x})$
10. Find two positive numbers whose sum is 12 and their product is maximum.
11. Find two positive numbers whose product is 20 and their sum is minimum.
12. Find the smallest possible value of $x^2 + y^2$ given that $x + y = 10$.

III. Answer the Following (5 Marks)

1. A stone is dropped into a pond causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate at 2 cm per second. When the radius is 5 cm find the rate of changing of the total area of the disturbed water?
2. A beacon makes one revolution every 10 seconds. It is located on a ship which is anchored 5 km from a straight shore line. How fast is the beam moving along the shore line when it makes an angle of 45° with the shore?
3. A conical water tank with vertex down of 12 metres height has a radius of 5 metres at the top. If water flows into the tank at a rate 10 cubic m/min, how fast is the depth of the water increases when the water is 8 metres deep?

4. A ladder 17 metre long is leaning against the wall. The base of the ladder is pulled away from the wall at a rate of 5 m/s. When the base of the ladder is 8 metres from the wall,

   (i) how fast is the top of the ladder moving down the wall?

   (ii) at what rate, the area of the triangle formed by the ladder, wall, and the floor, is changing?

5. A police jeep, approaching an orthogonal intersection from the northern direction, is chasing a speeding car that has turned and moving straight east. When the jeep is 0.6 km north of the intersection and the car is 0.8 km to the east. The police determine with a radar that the distance between them and the car is increasing at 20 km/hr. If the jeep is moving at 60 km/hr at the instant of measurement, what is the speed of the car?

6. Find the equation of tangent and normal to the curve given by $x = 7\cos t$ and $y = 2\sin t$, $t \in \mathbb{R}$ at any point on the curve.

7. Find the angle between the rectangular hyperbola $xy = 2$ and the parabola $x^2 y + 4 = 0$.

8. Show that the two curves $x^2 - y^2 = r^2$ and $xy = c^2$ where c, r are constants, cut orthogonally.

9. Show that the value in the conclusion of the mean value theorem for

   (i) $f(x) = \frac{1}{4}$ on a closed interval of positive numbers [a, b] is $\sqrt{ab}$

   (ii) $f(x) = Ax^2 + Bx + C$ on any interval [a, b] is $\frac{a+b}{2}$

10. A race car driver is kilometer stone 20. If his speed never exceeds 150 km/hr, what is the maximum kilometer he can reach in the next two hours.

11. Suppose that for a function f (x), $f'(x) \le 1$ for all $1 \le x \le 4$. Show that $f(4) - f(1) \le 3$.

12. Write the Maclaurin series expansion of the following functions:

    (i) e^x (ii) $\sin x$ (iii) $\cos x$ (iv) $\tan^{-1}(x)$; $-1 \le x \le 1$

13. Evaluate the following limits, if necessary use l'Hôpital Rule :

    (i) $\lim_{x \to 0^+} x^x$ (ii) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$ (iii) $\lim_{x \to 0^+} (\cos x)^{\frac{1}{x^2}}$

14. For the function $f(x) = 4x^3 + 3x^2 - 6x + 1$ find the intervals of monotonicity, local extrema, intervals of concavity and points of inflection.

15. A rectangular page is to contain 24 cm² of print. The margins at the top and bottom of the page are 1.5 cm and the margins at other sides of the page is 1 cm. What should be the dimensions of the page so that the area of the paper used is minimum.

16. A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 1,80,000 sq.mtrs in order to provide enough grass for herds. No fencing is needed along the river. What is the length of the minimum needed fencing material?

17. Find the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius 10 cm.

18. Prove that among all the rectangles of the given perimeter, the square has the maximum area.

19. Find the dimensions of the largest rectangle that can be inscribed in a semi circle of radius r cm.

20. A manufacturer wants to design an open box having a square base and a surface area of 108 sq.cm. Determine the dimensions of the box for the maximum volume.